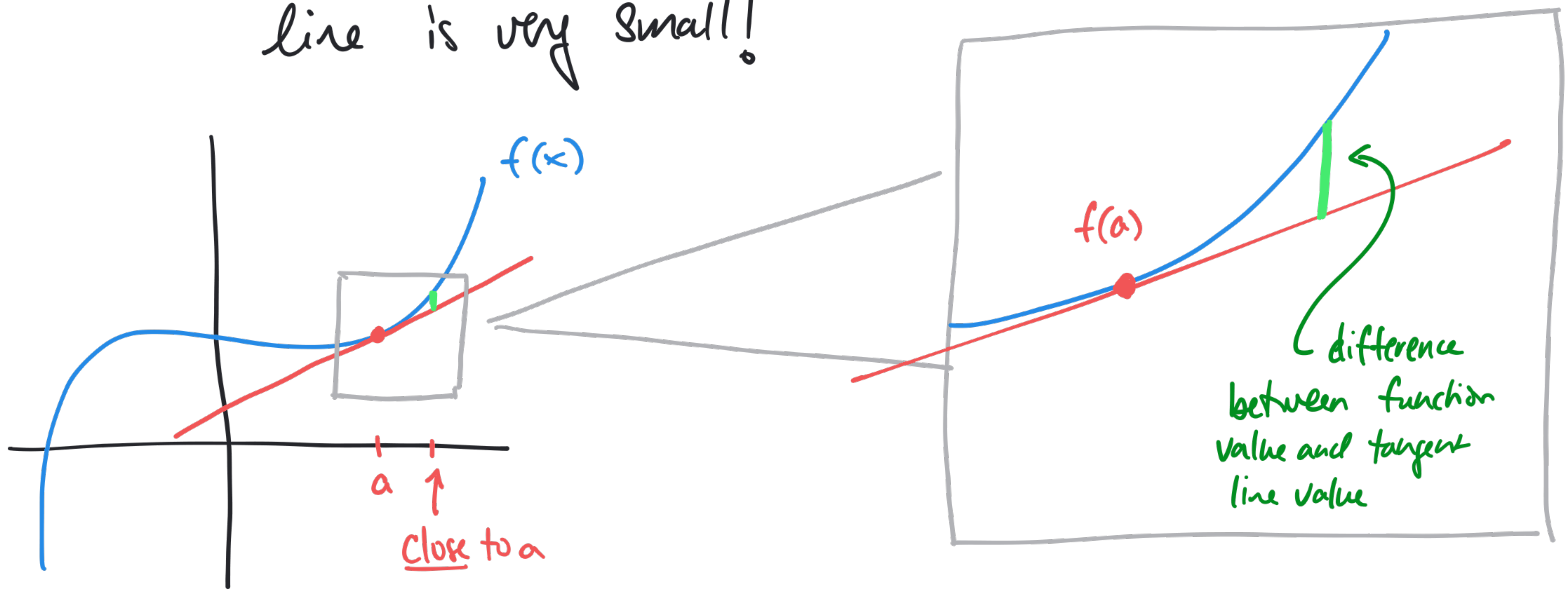
# Intro Video: Section 3.10
# Tangent Line Approximation / Linear approximation / Linearization / Differentials

Math F251X: Calculus I

Idea: Near a point $x=a$, the difference between the value of the function and the value of the tangent line is very small!




Name for the tangent line to $f(x)$ at $x=a$:

The linearization of $f(x)$ at $x=a$ is the function

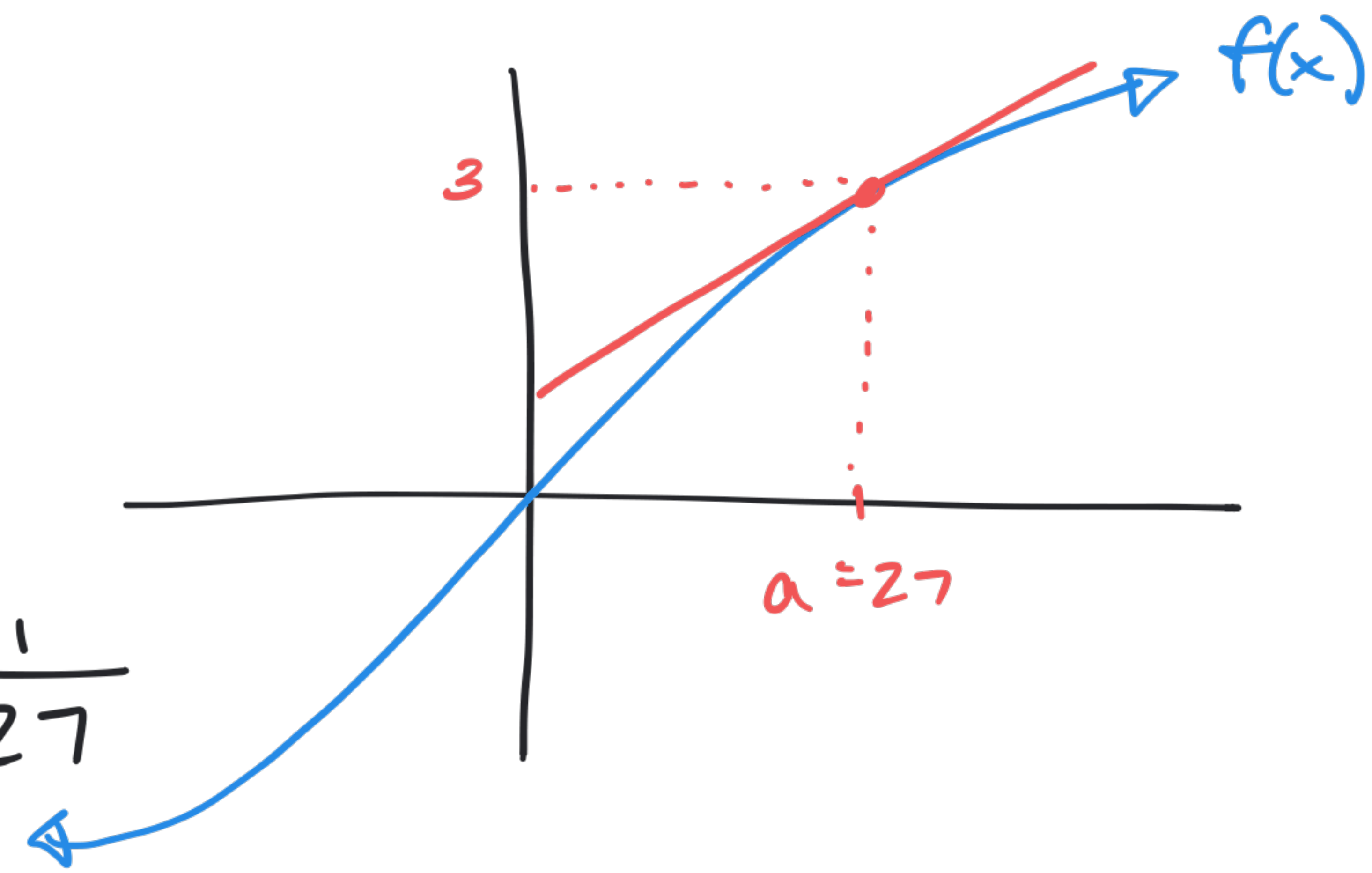
$$\boxed{L(x) = f'(a)(x-a) + f(a)}$$

The line with slope $f'(a)$ passing through the point of tangency

Example: $f(x) = x^{1/3}$. Use linearization/tangent line approximation to approximate the value of $f(26.9)$.

$\to$ That is, estimate $\sqrt[3]{26.9}$

Know $f(27) = 3$




$f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow$

$f'(27) = \frac{1}{3(27)^{2/3}} = \frac{1}{3(27^{1/3})^2} = \frac{1}{27}$

$L(x) = \frac{1}{27}(x - 27) + 3 \Rightarrow L(26.9) = L(27 - \frac{1}{10}) =$

$\frac{1}{27}(\cancel{27} - \frac{1}{10} - \cancel{27}) + 3 = \frac{-1}{270} + 3 = 2.9963$

By computer, $f(26.9) = \sqrt[3]{26.9} = 2.99629$

Error $= L(26.9) - f(26.9) = .00000458 \leftarrow 4.5 \times 10^{-6}$

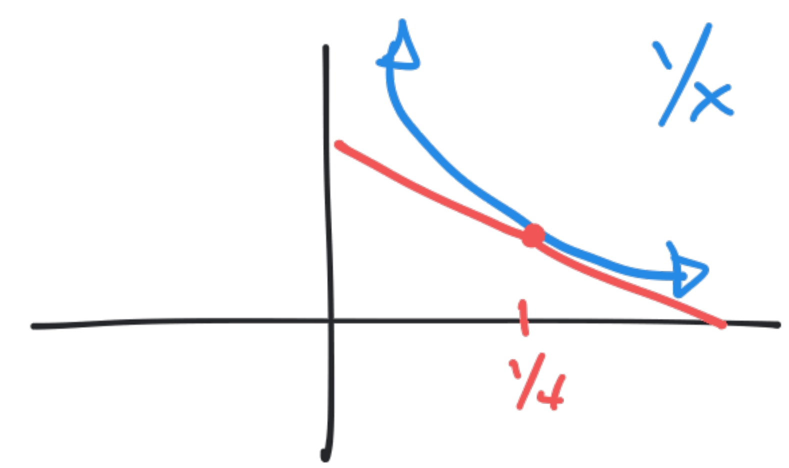
Tangent line approximation summary: to approximate $f(x)$ at $x=b$:

1) Identify a point $a$ near $b$ where it's easy to evaluate the function

2) Construct the linearization of $f$ at $a$, where

$$L(x) = f'(a)(x-a) + f(a)$$ is the tangent line to $f$ at $a$

3) Evaluate $L(x)$ at $x=b$




Example: Estimate $\frac{1}{4.002}$.

$f(x) = \frac{1}{x}$, $a = 4$ (since $f(4) = \frac{1}{4} = 0.25$)

$f'(x) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2}$ so $f'(4) = \frac{-1}{4^2} = \frac{-1}{16}$

So linearization of $f$ at $x=4$ is $L(x) = \frac{-1}{16}(x-4) + \frac{1}{4}$

and $L(4.002) = L\left(4 + \frac{2}{1000}\right) = \frac{-1}{\cancel{16}_{8}}\left(\frac{\cancel{2}}{1000}\right) + \frac{1}{4} = \frac{1}{4} - \frac{1}{8000} \approx 0.249875$

$f(4.002) = 0.2498750624\ldots$

# Differentials

Idea: $\dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx} \Rightarrow \Delta y = \dfrac{dy}{dx} \cdot \Delta x \Rightarrow \Delta y \approx f'(x)\Delta x$

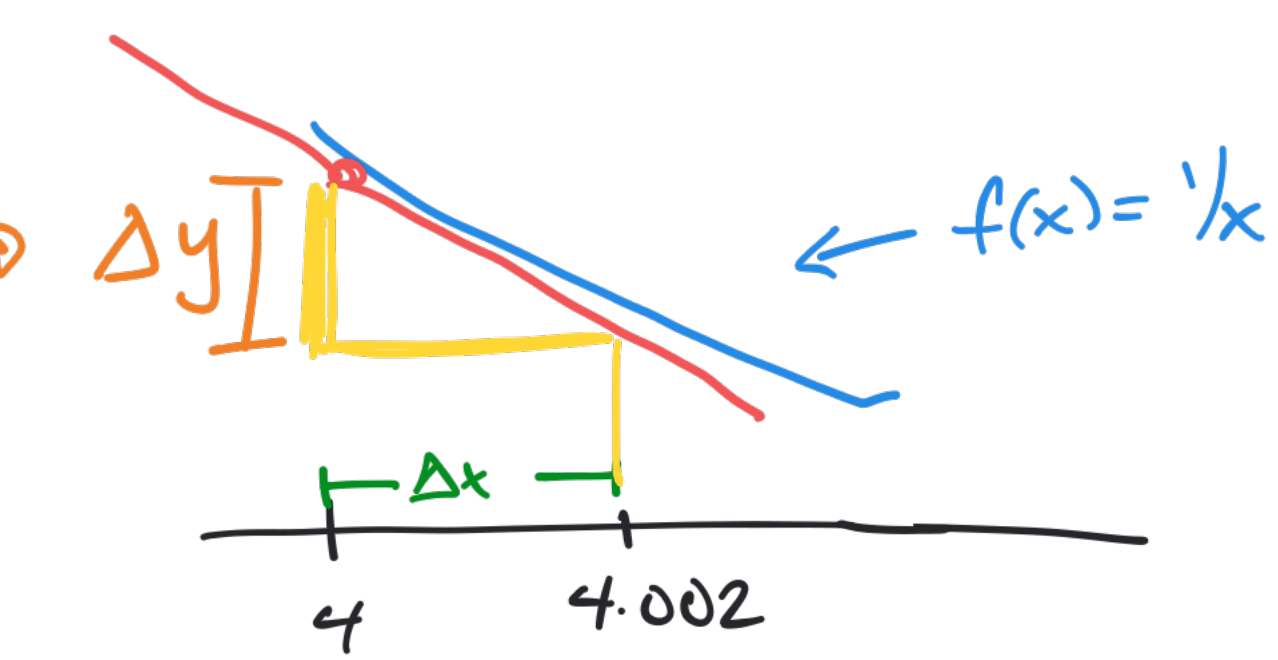
**Example**: What is $\Delta y$ in the estimate of $\dfrac{1}{4.002}$?

$\dfrac{\Delta y}{\Delta x} \approx f'(x)$ where $f(x) = \dfrac{1}{x} \Rightarrow f'(x) = \dfrac{-1}{x^2}$

$\Delta y = f'(4)\,\Delta x$ Here $\Delta x = 0.002 = \dfrac{2}{1000} = \dfrac{1}{500}$

So $\Delta y = \dfrac{-1}{16} \cdot \dfrac{1}{500} = \dfrac{-1}{8000} = -0.000125$

$\Delta y \approx -0.000125$

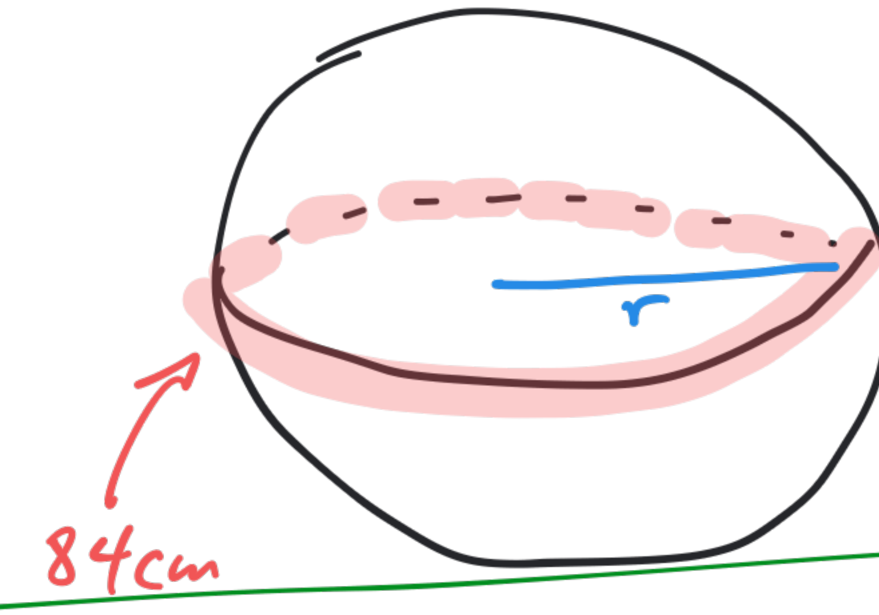


Example: The circumference of a sphere was measured to be 84 cm with a possible error of $\pm\frac{1}{2}$ cm.

→ What is the maximum error in the surface area? Relative error?

Recall: circumference $= 2\pi r \Rightarrow r = \frac{84}{2\pi}$

$SA = 4\pi r^2$




$\Delta \text{circ} = \pm\frac{1}{2}\text{ cm} \Rightarrow \Delta r = \frac{\pm 1}{4\pi}$

$$\Delta SA \approx \left( \underbrace{8\pi r}_{\frac{\Delta SA}{\Delta r}} \Big|_{r=\frac{84}{2\pi}} \right) \Delta r \Rightarrow$$

When circ $= 84$, surface area

$$= 4\pi\left(\frac{84}{2\pi}\right)^2 = \frac{\cancel{4\pi}\cdot(84)^2}{\cancel{4}\pi^{\cancel{2}}} = \frac{84^2}{\pi}$$

$$\Delta SA \approx \cancel{8}\cancel{\pi}\left(\frac{84}{\cancel{2}\cancel{\pi}}\right)\left(\frac{\pm 1}{\cancel{4}\pi}\right) = \pm\frac{84}{\pi} \approx 26.7\text{ cm}^2$$

$$\text{Relative error} = \frac{\text{error in SA}}{\text{total SA}} = \frac{\pm 84/\pi}{(84)^2/\pi} = \pm\frac{1}{84} \approx 0.012$$